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$\mathcal{N} = 1$ conformal supergravity and twistor-string theory

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Abstract: The physical states of $\mathcal{N} = 4$ conformal supergravity in four dimensions occur in twistor-string theory by Berkovits and Witten ([1]). We study two alternative versions of twistor-string theory based on the B-model of weighted projective space $\mathbf{WCP}^{3|2}$ and based on a certain construction involving open strings. The spacetime fields described by the twistor superfields contain the physical states of $\mathcal{N} = 1$ conformal supergravity from above $\mathcal{N} = 4$ superspace approach.

Keywords: Superstrings and Heterotic Strings, Topological Strings, Conformal and W Symmetry.

Contents

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1. Introduction

Contrary to the fact that the usual string theories are not conformally invariant in the target space(providing Einstein supergravity), in twistor-string theory [2] due to the superconformal invariance in spacetime the conformal supergravity arises. In four dimensions, the action has a quadratic Weyl tensor. See [3] for the review of conformal supergravity.

In [1], one of the main results was that the gauge-singlet sector of twistor-string theory has the same physical states as $\mathcal{N}=4$ conformal supergravity in four dimensions. The $\mathcal{N}=4$ chiral superfield which is a function of a chiral Minkowski superspace with coordinates coincides with a coupling of the abelian gauge g-field to the boundary of an open string world sheet. This boundary corresponds to a point in Minkowski spacetime and is defined via the twistor equations [2]. To determine the helicities of massless state in Minkowski spacetime, the relation [4, 5] between the degree of homogeneous coordinates of twistor space and the helicity as well as the gauge invariance and constraint were used. In the conformal supergravity side, the analysis for the solutions to the field equations which have higher derivatives(the spin 3/2 field has third-order derivatives and the spin 2 field has fourth-order derivatives) was used. The occurrence of two states with the same helicity where they form a doublet reflects the nonunitarity of the theory.

In this paper, by simply replacing $\mathbf{CP}^{3|4}$ with the weighted projective space $\mathbf{WCP}^{3|2}$ introduced in [2], we study how their physical states correspond to each other. In the twistor-string theory side, this weighted projective space has different weights in the fermionic coordinates and its bosonic submanifold is the same as $\mathbf{CP}^3$ of $\mathbf{CP}^{3|4}$. The only bosonic and fermionic homogeneous coordinates of weight one participate in $\mathcal{N}=1$ supersymmetry. In the $\mathcal{N}\leq 4$ conformal supergravity side, the number of spinor index of chiral field strength superfield depends on the number of supersymmetry $\mathcal{N}$ via $2s=4-\mathcal{N}$. When $\mathcal{N}\neq 4$, there exists a chiral superfield which has nonzero spinor indices. In particular, when $\mathcal{N}=1$, the theory can be described by a chiral superfield(which is a function of a chiral Minkowski superspace) with three spinor indices. By realizing that this $\mathcal{N}=1$ chiral superfield can be obtained from above $\mathcal{N}=4$ chiral superfield(by taking three superspace derivatives that do not play a role in $\mathcal{N}=1$ superspace and putting the other three fermionic coordinates to zero), we study the relation between the spacetime fields described by the twistor superfields and the physical states of $\mathcal{N}=1$ conformal supergravity from the $\mathcal{N}=4$ superspace approach [1].

In section 2, we describe the twistor-string vertex operators in two alternative versions. In section 3, we discuss the spectrum of massless fields in Minkowski spacetime characterized by twistor fields. In section 4, we consider the linearized spectrum of conformal supergravity and compare it with the twistor-string theory results of previous section. In doing this, the role of $\mathcal{N}=1$ chiral superfield in $\mathcal{N}=4$ superspace is crucial. In section 5, we make some remarks after summarizing the main results of this paper.

2. Vertex operator of the B-model on $\mathbf{WCP}^{3|2}$

Let us describe the twistor-string vertex operators in the open string version first and then we consider the topological B-model of $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ which is denoted by $\mathbf{WCP}^{3|2}$ in this paper, for simplicity.

In the open twistorial string theory [6], the worldsheet action can be written as the homogeneous coordinates(or super-twistor variables)

$$Z^I = (\lambda^a, \mu^{\dot{a}}, \psi^A)\,, \qquad a, \dot{a}, A = 1,2$$

of $\mathbf{WCP}^{3|2}$ and its complex conjugates $\overline{Z}^I$, conjugate super-twistor variables Y_I and its complex conjugates $\overline{Y}_I$, a worldsheet GL(1) connection through the covariant derivatives plus the action for current algebra. The conformal dimensions of $(Z^I, \overline{Z}^I, Y_I, \overline{Y}_I)$ are $(0,0), (0,0), (1,0)$, and $(0,1)$ respectively. The physical states are described by the dimension one vertex operators that are GL(1)-neutral and primary fields under the Virasoro and GL(1) generators. The boundary condition for open string implies that open string vertex operators can be written in terms of Z^I and Y_I and current algebra variables(not $\overline{Z}^I$ and $\overline{Y}_I$).

The simplest dimension zero primary fields are any function $\phi(Z^I)$(that is GL(1)-neutral, that is, invariant under GL(1)) on $\mathbf{WCP}^{3|2}$. By combining this with any current j_r where $r=1,2,\ldots,\mathrm{dim}G$ of the current algebra associated with some group G, the Yang-Mills vertex operator of dimension 1 is given by $j_r\phi^r(Z)$ [6].

On the other hand, the conformal supergravity multiplet can be described by dimension 1 vertex operator by using either Y_I or ∂Z^I(recall that the dimension of these is 1). Then one can construct them as follows:

$$Y_I f^I(Z)\,, \qquad g_I(Z)\partial Z^I\,.$$

These are GL(1)-invariant if f^I carries GL(1) charge 1 for the indices $a, \dot{a}, A=1$ and 3 for the index $A=2$. In other words, under the $Z^I \to tZ^I$ where Z^I stands for the homogeneous bosonic and fermionic coordinates of weight 1, the f^I scales as $f^I \to tf^I$. The corresponding Y_I has GL(1) charge -1. Under the $Z^I \to t^3 Z^I$ where Z^I stands for the homogeneous fermionic coordinate of weight 3, the f^I scales as

$$f^{A=2} \rightarrow t^3 f^{A=2}.$$

The corresponding Y_I has GL(1) charge -3.

Similarly the vertex operator corresponding to g_I is GL(1)-invariant if g_I carries GL(1) charge -1 for the indices $a, \dot{a}, A = 1$ and -3 for the index $A = 2$. In other words, under the $Z^I \rightarrow tZ^I$ where Z^I stands for the homogeneous bosonic and fermionic coordinates of weight 1, the g_I scales as $g_I \rightarrow t^{-1} g_I$. Under the $Z^I \rightarrow t^3 Z^I$ where Z^I stands for the homogeneous fermionic coordinate of weight 3, the g_I scales as

$$g_{A=2} \rightarrow t^{-3} g_{A=2}.$$

The constraint $\partial_I f^I = 0$ coming from the condition of primary field on the vertex operator implies that the f^I is a volume-preserving vector field. The other constraint $g_I Z^I = 0$ implies that $g_I dZ^I$ is well-defined one form.

Let us consider the second version of twistor-string theory and we denote the complex homogeneous coordinates of $\mathbf{WCP}^{3|2}$ as $Z^I = (\lambda^a, \mu^{\dot{a}}, \psi^A)$ where λ^a and $\mu^{\dot{a}}$ are bosonic spinors of opposite helicity and ψ^A are fermions. Let us consider the conformal supergravity sector. In the B-model, closed string modes describe the nontrivial deformations of the complex structure of the target space $\mathbf{WCP}'^{3|2}$ which denotes the region in $\mathbf{WCP}^{3|2}$ in which the λ^a are not both zero. The deformation should preserve the holomorphic volume-form Ω. We cover $\mathbf{WCP}'^{3|2}$ with open sets U_i [7]. One can take two open sets, a set U_1 characterized by $\lambda^1 \neq 0$ and a set U_2 characterized by $\lambda^2 \neq 0$(Note that $\mathbf{WCP}^{3|2}$ is not deformable and rigid). On their intersections U_{ij} one glues the open sets U_i through diffeomorphism of the form $Z^I \rightarrow Z^I + \epsilon f^I_{ij}$ where ϵ is an infinitesimal parameter and $f_{ji} = -f_{ij}$.

One can proceed the discussion of [1] for our weighted projective space and, for example, the nonlinear action can be written similarly as [1] and is given by

$$\int_{\mathbf{WCP}^{3|2}} d\overline{X}^{\overline{I}} d\overline{X}^{\overline{J}} d\overline{X}^{\overline{K}} b_{\overline{I}I} N^I_{\overline{J}\overline{K}}\, \Omega$$

where X^I are local complex coordinates on $\mathbf{WCP}^{3|2}$ and the complex conjugates $\overline{X}^I$ are bosonic. The Ω is holomorphic volume-form or measure. From an almost complex structure, one can construct an invariant tensor called Nijenhuis tensor N. In Minkowski spacetime, the condition for the vanishing of Nijenhuis tensor(coming from the equation of motion b) corresponds to $F_{abcd} = 0$ which is the self-dual part of Weyl tensor and is symmetric in all their indices, according to the result of Penrose [8].

Then the whole contribution of spacetime description of above action plus a G^2 interaction (G is a spin 2-field and part of the spacetime description of the twistor field b) [2] gives rise to $\int d^4x \sqrt{g}(G^{abcd} F_{abcd} - \frac{1}{2}\epsilon G^2)$ which is equal to $\frac{1}{2\epsilon} \int d^4x \sqrt{g} F^{abcd} F_{abcd}$ after integrating out G. A degree one instanton is a copy of $\mathbf{CP}^1$ embedded in $\mathbf{WCP}^{3|2}$ and in the Penrose transform, this corresponds to a point in a chiral Minkowski superspace. One can define a function by an integral of b over the curve with moduli x and θ_1. Then the θ_1-expansion of this function $\mathcal{W}_{abc}^{\mathcal{N}=1}(x,\theta_1)$ is given by $\mathcal{W}_{abc}^{\mathcal{N}=1}(x,\theta_1) = \Phi_{abc}(x) + \theta_1^d G_{abcd}(x) + \theta_1^d \theta_{1d} \Delta_{abc}(x)$ where the expression for the effective action quadratic in b over the moduli space with an appropriate measure reads $\int d^4 x^{a\dot{a}} d^2 \theta_1^e [\mathcal{W}_{bcd}^{\mathcal{N}=1}(x,\theta_1)]^2$. By substituting $\mathcal{W}_{abc}^{\mathcal{N}=1}(x,\theta_1)$ into this action, the above G^2 interaction can be obtained.

3. Spectrum of massless fields in Minkowski spacetime

Let us consider the twistor field $f^I(Z)$ which is a function of four bosonic and two fermionic variables $\lambda^a, \mu^{\dot{a}}(a, \dot{a} = 1,2)$ and $\psi^A(A = 1,2)$. Let us denote $\psi^{A=1}$ as ψ and $\psi^{A=2}$ as χ

$$\psi^{A=1} \equiv \psi\,, \qquad \psi^{A=2} \equiv \chi\,.$$

Since the $f^I(Z)$, for each I except an index $A = 2$, is homogeneous in Z^I of degree 1, there exist four bosonic and one fermionic helicity states, each of helicity 3/2(since a massless state in Minkowski spacetime has a helicity $1 + \frac{1}{2}$deg. [4, 5]) when we put $\psi^A = 0$ and do not take the spinor index. For $A = 2$, the $f^{A=2}(Z)$ is homogeneous in Z^I of degree 3 and there is one fermionic helicity state 5/2 by same reason. Both spinor index a and $\dot{a}$ give a helicity 1/2 and $-1/2$. Then this leads to two bosonic states of helicity 2 and two of helicity 1. Moreover there are two states of helicity 3/2 and 5/2. According to the arguments of [1], after taking account of the gauge invariance and the constraint, there are two bosonic states of helicity 2(two bosonic states of helicity 1 are removed) and two fermionic states of helicity 3/2 and 5/2.

Let us recall [2] that the action of $\mathrm{SU}(2,2|1)$ on the the homogeneous coordinates (Z^I, ψ) is generated by 5×5 matrices which are supertraceless. The bosonic conformal algebra $\mathrm{SU}(2,2)$ which is the covering group for $\mathrm{SO}(4,2)$ is represented by the $15 (= 4^2 - 1 = \frac{6\times 5}{2})$ matrices which live in the 4×4 left-upper part. A bosonic generator for chiral $\mathrm{U}(1)_R$ transformation is represented by a diagonal matrix with entries $\mathrm{diag}(1,1,1,1,4)$ up to an overall scale. Moreover a spinorial(supersymmetry) generator and a special conformal supersymmetry generator are realized as the matrices with fifth row and fifth column. All 5×5 matrices are supertraceless where the trace of the 4×4 left-upper part is equal to the $(5,5)$ matrix element.

Now for nonzero ψ^A, one can expand $f^I(Z)$ in powers of ψ and χ:

$$f^I(\lambda, \mu, \psi, \chi) = f_0^I(\lambda,\mu) + f_{1\psi}^I(\lambda,\mu)\psi + f_{1\chi}^I(\lambda,\mu)\chi + f_{2\psi\chi}^I(\lambda,\mu)\psi\chi\,,$$

where $f_{1\psi}^I$ is homogeneous in (λ,μ) with degree 0 and provides a massless state of helicity 1 when we ignore the angular momentum carried by the index I, as observed above, $f_{1\chi}^I$ is homogeneous in (λ,μ) with degree -2(leading to a helicity 0) and $f_{2\psi\chi}^I$ is homogeneous in (λ,μ) with degree -3(giving a helicity $-1/2$). Here the index I denotes by $a, \dot{a}$ or $A = 1$.

Finally, we list the full structure of helicity states described by the field $f^I(Z)$ by taking into account of angular momentum, the gauge invariance and the constraint

$$\begin{aligned}
\lambda^a f_a(Z) &: (2,0)\,, \quad \left(\frac{3}{2},-1\right), \\
\mu^{\dot a} f_{\dot a}(Z) &: (2,0)\,, \quad \left(\frac{3}{2},-1\right), \\
f^{A=1}(Z) &: \left(\frac{3}{2},1\right), \quad (1,0)\,, \quad (0,0)\,, \quad \left(-\frac{1}{2},-1\right),
\end{aligned} \tag{3.1}$$

where the first element is the helicity and the second element is the $\mathrm{U}(1)_R$ representation. The $\mathrm{U}(1)_R$ transformation laws of the fields are determined by the chiral weight [3]. Compared with the spectrum of $\mathcal{N}=4$ field contents, the first two entries in each $f^I(Z)$ are exactly a truncation of $\mathcal{N}=4$ spectrum. They have the same helicity and $\mathrm{U}(1)_R$ charge. The state $(2,\mathbf{1})$ of $\mathcal{N}=4$ twistor field has $\mathrm{U}(1)_R$ charge 0 and the state $(\frac{3}{2},\overline{\mathbf{4}})$ of $\mathcal{N}=4$ twistor field has $\mathrm{U}(1)_R$ charge -1 and so on. Since the helicity 5/2 of fermionic state characterized by $f^{A=2}(Z)$ is greater than 2, this field is not allowed in the conformal supergravity.[1] Note that $f^I(Z)$ where $I=a$ or $I=\dot a$ has no dependence on χ of weight 3 in $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ and is a 'truncation' of the $\mathcal{N}=4$ case and $f^{A=1}(Z)$ has a dependence on χ. Remember that for $\mathcal{N}=4$ theory, all the $\psi^A(A=1,2,3,4)$ play the role of the fermionic coordinates in the chiral Minkowski superspace through twistor equations.

Let us consider the twistor field $g_I(Z)$ and the Lorentz scalars $(\lambda^a g_a, \mu^{\dot a} g_{\dot a}, \partial_a g^a, \partial_{\dot a} g^{\dot a})$ are homogeneous of degree $(0,0,-2,-2)$ since g_a and $g_{\dot a}$ are homogeneous of degree -1. By counting of the gauge invariance and the constraint(leading to the removal of two fields of degree 0), there exist two twistor fields of degree -2 leading to two massless states of helicity 0. The field $g_{A=1}$ is homogeneous of weight -1 describing massless states of helicity $\frac{1}{2}$ while the field $g_{A=2}$ is homogeneous of weight -3 describing massless states of helicity $-\frac{1}{2}$. However, this is not allowed in our consideration. Then the complete structure of helicity states by the field $g_I(Z)$ are summarized by

$$\begin{aligned}
\partial_a g^a(Z) &: (0,1)\,, \quad \left(-\frac{1}{2},0\right), \\
\partial_{\dot a} g^{\dot a}(Z) &: (0,1)\,, \quad \left(-\frac{1}{2},0\right), \\
g_{A=1}(Z) &: \left(\frac{1}{2},1\right), \quad (0,0)\,, \quad (-1,0)\,, \quad \left(-\frac{3}{2},-1\right).
\end{aligned} \tag{3.2}$$

[1]The possibility of having a spin $s>2$ conformal supergravity field has been discussed in the existence of some nontrivial $\mathcal{N}>4$ conformal supergravity [9]. If this theory can be coupled to Poincare supergravity, then these higher spin fields occur in the conformal supergravity action like $\overline{\zeta}_{\frac{5}{2}}\gamma\cdot\partial\Box^2\zeta_{\frac{5}{2}}$. It was difficult to form massive higher multiplets in the spectra of the combined conformal supergravity and Poincare supergravity system. Even if $\mathcal{N}>4$ conformal supergravity with higher spin fields can exist, they cannot be coupled to the corresponding Poincare supergravity consistently. The appearance of scalar W_{ABCD} [10] for $\mathcal{N}>4$ restricts some constraint on this and corresponding conformally invariant action which is not known differs from the one in chiral superspace because this scalar is not chiral. It is not clear how to construct the 'full' action including higher spins with higher power of derivatives.

Although in $\mathcal{N} = 4$ case, the massless fields described by $g_I(Z)$ possess the opposite helicities and conjugate SU(4) representations from those described by $f^I(Z)$, this is not true for our $\mathcal{N} = 1$ case. We also note that the first two entries in each $g_I(Z)$ have the same helicities as the one in $\mathcal{N} = 4$ case [1] but the $\mathrm{U}(1)_R$ charges are different. By acting some quantity on $g^I(Z)$ of $\mathcal{N} = 4$ twistor field($I = a$ or $\dot{a}$) which will change the $\mathrm{U}(1)_R$ charges of 4 of state $(0, \mathbf{1})$ and 3 of state $(-\frac{1}{2}, \overline{\mathbf{4}})$ into 1 or 0 respectively, this modified $g_I(Z)$ will give the above assignments. Similarly the modified $g_{A=1}(Z)$ that can be obtained by acting some quantity on $g_{A=1}(Z)$ will change $\mathrm{U}(1)_R$ charges 3 of state $(\frac{1}{2}, \overline{\mathbf{4}})$ and 2 of state $(0, \overline{\mathbf{10}} \oplus \mathbf{6})$ into 1 or 0 respectively. Since the fermionic coordinates θ^A_a has $\mathrm{U}(1)_R$ charge 1, it is natural to consider a superspace derivative which has $\mathrm{U}(1)_R$ charge -1, as a multiplicative factor.

We will verify that the spacetime fields characterized by the twistor superfields $f^I(Z)$ and $g_I(Z)$ contain the physical states of $\mathcal{N} = 1$ conformal supergravity in section 4.

4. Linearized spectrum of $\mathcal{N} = 1$ conformal supergravity

The linearized $\mathcal{N} \leq 4$ conformal supergravity [10]–[12] can be described by a chiral field strength superfield which has $2s$ spinor indices and has dimension $s = \frac{1}{2}(4-\mathcal{N})$. It contains spins, $s, s+\frac{1}{2}, \ldots, s+\frac{\mathcal{N}}{2}(=2)$ and satisfies some constraints. For example, for $\mathcal{N} = 4$, the chiral field strength superfield with no spinor indices has the following component field expansion [1]

$$\begin{aligned}\mathcal{W}^{\mathcal{N}=4}(x,\theta) = {} & \ldots + (\theta^3)^{(abc)}_D \, (\partial\eta)^D_{(abc)} + (\theta^3)^{aC}_{[AB]} \, \xi^{[AB]}_{aC} + \\ & + (\theta^4)^{A(ab)}_B \, (\partial V)^B_{(ab)A} + (\theta^4)^{(abcd)} \, W_{abcd} + (\theta^4)^{[AB]}_{[CD]} \, d^{[CD]}_{[AB]} + \\ & + (\theta^5)^{a[AB]}_C \, \partial_{a\dot{a}} \overline{\xi}^{\dot{a}C}_{[AB]} + (\theta^5)^{A(abc)} \, (\partial\rho)_{A(abc)} + \cdots ,\end{aligned} \tag{4.1}$$

where the terms with lower θ's than θ^3 and the terms higher θ's than θ^5 are not written explicitly. The flat superspace covariant derivatives are given by

$$D^a_A = \frac{\partial}{\partial\theta^A_a} + \overline{\theta}^{\dot{a}}_A \partial_{a\dot{a}} , \qquad \overline{D}^A_{\dot{a}} = \frac{\partial}{\partial\overline{\theta}^{\dot{a}}_A} , \qquad a, \dot{a} = 1,2 \qquad A = 1,2,3,4 .$$

The chirality of $\mathcal{W}^{\mathcal{N}=4}(x,\theta,\overline{\theta})$ implies $\overline{D}^A_{\dot{a}} \mathcal{W}^{\mathcal{N}=4}(x,\theta,\overline{\theta}) = 0$ and this leads to an independence of $\overline{\theta}^{\dot{a}}_A$. In (4.1), we only considered θ^A_a-expansion:chiral Minkowski superspace.

Let us consider $\mathcal{N} = 1$ chiral field strength superfield characterized by three(that is, $2s = 4 - \mathcal{N} = 3$) spinor indices, according to above counting, and describe it in terms of $\mathcal{N} = 4$ chiral superfield $\mathcal{W}^{\mathcal{N}=4}(x,\theta)$ because one can use the results of [1] directly once we find out the exact relation between the two chiral superfields. One can construct the following quantity where the spinor indices a, b and c can be obtained from three superspace derivatives with contracted SU(4) indices(this is equivalent to take three superspace derivatives which are irrelevant to $\mathcal{N} = 1$ superspace) by using the 4th rank epsilon tensor and put the other fermionic coordinates to zero after differentiation:

$$\mathcal{W}^{\mathcal{N}=1}_{abc}(x,\theta_1) = \epsilon_{1ABC} D^A_a D^B_b D^C_c \mathcal{W}^{\mathcal{N}=4}(x,\theta)\Big|_{\theta_2=\theta_3=\theta_4=0} \tag{4.2}$$

which is symmetric in the indices a, b and c: An interchanging of any two superspace derivatives has minus sign due to the anticommutativity and minus sign for the changing of SU(4) indices with an antisymmetric epsilon tensor.

Let us compute the right hand side of (4.2) by acting the superspace derivatives on (4.1) and putting $\theta_i (i = 2, 3, 4)$ to zero. One can easily see that the terms which are not written explicitly in (4.1) do not contribute in this procedure. When three D's which has the SU(4) indices 2, 3 and 4 are acting on the object which contains more than six θ's, we are left with an object with more than three θ's and those θ's have θ_i where $i \neq 1$. By putting $\theta_i (i = 2, 3, 4)$ to zero, the results do not have any contributions. On the other hand, when three D's are acting on the object which has two θ's at most, there exists a single D we have not used. Therefore, there are no contributions. Note that the second term of D_A^a above does not contribute at all when acting on some function appearing in the $\mathcal{W}^{\mathcal{N}=4}(x, \theta)$ because it contains $\overline{\theta}_A^{\dot{a}}$ and at the final expression we put this to zero also.

Now let us first consider the cubic term in θ transforming **4** of SU(4) with $(\partial\eta)_{(abc)}^D$. One can write explicitly as follows

$$\epsilon_{1ABC} D_a^A D_b^B D_c^C \, (\theta^3)^{D(efg)} = \epsilon_{1ABC} \epsilon^{DEFG} D_a^A D_b^B D_c^C \, \theta_E^e \theta_F^f \theta_G^g \, . \tag{4.3}$$

Then this will lead to the product of delta functions $\delta_1^D \delta_a^e \delta_b^f \delta_c^g$ and provides us with $(\partial\eta)_{(abc)}^{D=1}$. On the other hand, when three superspace derivatives act on $(\partial\eta)_{(abc)}^D$, since the second term of a superspace derivative D_A^a does not contribute at all, there is no contribution.

Similarly the next term in cubic θ can be written as, by applying the irreducible D-operators in $\mathcal{N} = 4$ superspace [13, 14] to the fermionic coordinates,

$$\epsilon_{1ABC} D_a^A D_b^B D_c^C \, (\theta^3)_{[DE]F}^d = \epsilon_{1ABC} \epsilon_{ef} D_a^A D_b^B D_c^C \, \theta_{[D}^d \theta_{E]}^e \theta_F^f \, ,$$

where the 2nd rank antisymmetric tensor ϵ_{ab} is defined in [15, 2]. The spinor indices can be raised or lowered by using this and its inverse ϵ^{ab}. By calculating the above quantity explicitly, it turns out there are six terms which vanish from the antisymmetric property of 4th rank and 2nd rank epsilon tensors.

Let us move on the quartic term in θ transforming **15** of SU(4):

$$\epsilon_{1ABC} D_a^A D_b^B D_c^C \, (\theta^4)_D^{E(de)} = \epsilon_{1ABC} \epsilon^{EFGH} \epsilon_{fg} D_a^A D_b^B D_c^C \, \theta_D^f \theta_F^g \theta_G^d \theta_H^e \, . \tag{4.4}$$

The contraction between 4th rank epsilon tensors gives rise to the product of two delta functions after acting three superspace derivatives on the θ's. When the index D and index E are equal to each other $D = E$, there exists a term like as $\epsilon_{cd} \theta_1^d (\partial V)_{(ab)A}^A$. However, since $V_{\mu A}^A = 0$ [1] that is nothing but irreducibility condition of SU(4) tensor, all these contributions become zero. When they are different from each other $D \neq E$, the SU(4) indices of a single remaining θ can be either 2,3 or 4. Therefore, these contributions are zero after we set $\theta_2 = \theta_3 = \theta_4 = 0$ eventually.

By writing the next term transforming a singlet **1** of SU(4) as

$$\epsilon_{1ABC} D_a^A D_b^B D_c^C \, (\theta^4)^{(defg)} = \epsilon_{1ABC} \epsilon^{DEFG} D_a^A D_b^B D_c^C \, \theta_D^d \theta_E^e \theta_F^f \theta_G^g \, , \tag{4.5}$$

then the θ that are left with contains SU(4) index 1 and the nontrivial result is given by $\theta_1^d W_{(abc)d}$.

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Moreover, the last term in quartic θ transforming $\mathbf{20'}$ of SU(4) can be written as

$$\epsilon_{1ABC} D_a^A D_b^B D_c^C \ (\theta^4)^{[DE]}_{[FG]} = \epsilon_{1ABC} \epsilon^{DEHI} \epsilon_{de} \epsilon_{fg} D_a^A D_b^B D_c^C \ \theta_H^d \theta_F^e \theta_I^f \theta_G^g \, .$$

In this case, twenty four terms are exactly cancelled each other by the antisymmetric property of epsilon tensors.

Now we turn to next order term in θ transforming $\mathbf{20}$ of SU(4) and decompose as follows:

$$\epsilon_{1ABC} D_a^A D_b^B D_c^C \ (\theta^5)^{[DE]}_{Fd} = \epsilon_{1ABC} \epsilon^{DEGH} D_a^A D_b^B D_c^C \ \left[(\theta^4)_{FGde} \theta_H^e \right] .$$

From the previous consideration (4.4), there is no contribution when the three superspace derivatives are acting on only the first factor $(\theta^4)_{FGde}$. Then the nontrivial contributions should contain a superspace derivative acting on the second factor θ_H^e. Of course, the remaining two superderivatives act on the first factor. Then the results can be written as the contractions between three 4th rank epsilon tensors together with the quadratic in θ's. It is easy to see that one of the SU(4) indices in θ's can be either 2,3 or 4. Therefore, there are no nonzero contributions after we put $\theta_i = 0$ where $i = 2, 3, 4$.

Let us consider the next term in fifth order in θ transforming $\mathbf{4}$ of SU(4) with $(\partial\rho)_{A(abc)}$ and write explicitly as follows

$$\epsilon_{1ABC} D_a^A D_b^B D_c^C \ (\theta^5)^{D(def)} = \epsilon_{1ABC} D_a^A D_b^B D_c^C \left[(\theta^4)^{DE(de)} \theta_E^f \right] . \tag{4.6}$$

One can do this similarly. When the SU(4) index D is equal to A, B or C among 36 terms, the index D can be $2, 3$ or 4 and one of the SU(4) indices in the remaining two θ's is equal to the index D. This implies that 18 terms do not contribute. However, when the index D appears in the remaining two θ's, then this will provide the following nonzero contribution $\epsilon_{de} \theta_1^d \theta_1^e (\partial\rho)_{1(abc)}$.

After we combine all the nonzero contributions (4.3), (4.5) and (4.6) together, the relation (4.2) reads

$$\begin{aligned} \mathcal{W}_{abc}^{\mathcal{N}=1}(x^{d\dot{d}}, \theta_d^1) &= (\partial\eta)_{(abc)} + \theta_1^d \left[C_{(abc)d} + \epsilon_{ab} (\partial A)_{cd} + \epsilon_{ac} (\partial A)_{db} + \epsilon_{ad} (\partial A)_{bc} \right] + \\ &\quad + \theta_1^d \theta_{d1} (\partial\rho)_{(abc)} \, , \end{aligned}$$

where the $W_{(abc)d}$ is decomposed as C_{abcd} which is symmetric in all indices plus A_{cd} which is symmetric in indices.[2] This is exactly the same as a superfield approach [12] to $\mathcal{N} = 1$ conformal supergravity where the complete analysis of the linearized $\mathcal{N} = 1$ conformal supergravity was given. What we have done so far is that for given $\mathcal{N} = 4$ superfield (4.1), we computed the right hand side of (4.2) and the right field contents of $\mathcal{N} = 1$ conformal supergravity were selected without imposing on the vanishing of extra field contents of $\mathcal{N} = 4$ theory by hand, by doing the superspace derivative algebra on the SU(4) tensorial structure of the fermionic coordinates.

[2] Here we use simplified notation as [1] and denote $(\partial A)_{(ab)} = \partial_\mu A_\nu (\sigma^{\mu\nu})_{(ab)}$, $(\partial\eta)_{(abc)} = \partial_\mu \eta_{\nu(a} (\sigma^{\mu\nu})_{bc)}$, $(\partial\rho)_{(abc)} = \partial_\mu \rho_{\nu(a} (\sigma^{\mu\nu})_{bc)}$, and $\rho_{\nu a} = \sigma^\mu_{a\dot{a}} \left(\partial_\mu \overline{\eta}_\nu^{\dot{a}} - \partial_\nu \overline{\eta}_\mu^{\dot{a}} + \frac{1}{2} \epsilon_{\mu\nu\tau\kappa} \partial^\tau \overline{\eta}^{\kappa\dot{a}} \right)$.

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The action for $\mathcal{N}=1$ conformal supergravity at the linearized level [12, 3] can be written as

$$S=\int d^4x\left(\int d^8\theta\,\mathcal{W}^2_{\mathcal{N}=4}\right)\Big|_{\theta_i=0}\rightarrow\int d^4x\int d^2\theta_1\,\left[\mathcal{W}^{\mathcal{N}=1}_{abc}(x,\theta_1)\right]^2.$$

By computing the θ_1-integrals with the explicit form for $\mathcal{W}^{\mathcal{N}=1}_{abc}(x^{d\dot{d}},\theta^1_d)$ above, this action leads to the sum of following component lagrangian

$$\mathcal{L}_1=(\partial A)_{(ab)}(\partial A)^{(ab)}\,,\qquad \mathcal{L}_{\frac{3}{2}}=(\partial\eta)_{(abc)}(\partial\rho)^{(abc)}\,,\qquad \mathcal{L}_2=C_{abcd}C^{abcd}\,.$$

The quartic and higher order fermionic terms can be found in [16]. See also recent review paper on the conformal supergravity [17]. The linearized $\mathcal{N}=1$ conformal supergravity can be described off-shell by a chiral field strength superfield $\mathcal{W}^{\mathcal{N}=1}_{abc}(x^{d\dot{d}},\theta_d,\overline{\theta}^{\dot{d}})$ that satisfies the constraint $\partial_{\dot{a}}{}^a D^b\mathcal{W}^{\mathcal{N}=1}_{abc}=\partial_a{}^{\dot{a}}\overline{D}^{\dot{b}}\overline{\mathcal{W}}^{\mathcal{N}=1}_{\dot{a}\dot{b}\dot{c}}$ [12, 3]. Then the condition for $\mathcal{W}^{\mathcal{N}=1}_{abc}(x,\theta,\overline{\theta})$ to be chiral, $\overline{D}^{\dot{d}}\mathcal{W}^{\mathcal{N}=1}_{abc}(x^{e\dot{e}},\theta_e,\overline{\theta}^{\dot{e}})=0$ implies that $\mathcal{W}^{\mathcal{N}=1}_{abc}(x^{e\dot{e}},\theta_e,\overline{\theta}^{\dot{e}})$ does not depend on $\overline{\theta}^{\dot{e}}$.

To determine the helicities for these fields, it is necessary to solve the higher derivative equations of motion: the spin 3/2-field has third-order derivatives and the spin 2-field has fourth-order derivatives.

For two gravitinos η^a_μ, the linearized equation of motion[3] is given by [1]

$$\partial^{e\dot{b}}\partial^{a(\dot{c}}\partial^{\dot{a})b}\sigma^{\mu}_{b\dot{b}}\eta_{\mu a}=0$$

and this implies

$$\eta_{\mu a}=\sigma^{b\dot{b}}_{\mu}\int d^4k\delta(k^2)e^{ik\cdot x}\left[\pi_a\pi_b\tau_{\dot{b}}\left(\eta_{-\frac{3}{2}}(k)+i\frac{x_0}{k_0}\eta'_{-\frac{3}{2}}(k)\right)+\pi_{(a}\widetilde{\tau}_{b)}\tau_{\dot{b}}\eta_{-\frac{1}{2}}(k)+\widetilde{\tau}_a\widetilde{\tau}_b\widetilde{\pi}_{\dot{b}}\eta_{\frac{3}{2}}(k)\right].\tag{4.7}$$

Here we used a pair of spinors $\pi,\widetilde{\pi}$ such that $k^{a\dot{a}}=\pi^a\widetilde{\pi}^{\dot{a}}$ and a pair of spinors $\tau,\widetilde{\tau}$ such that $\pi^a\widetilde{\tau}_a=1$ and $\widetilde{\pi}^{\dot{a}}\tau_{\dot{a}}=1$. Both $\eta_{-\frac{3}{2}}(k)$ and $\eta'_{-\frac{3}{2}}(k)$ which are independent fields combine to a 'dipole' field of helicity $-3/2$ and spacetime translations act in undiagonalizable reflecting the nonunitarity of the theory.

For A_ν, the equation of motion from the action is given by[4]

$$\partial_\mu\partial_\rho(\sigma^{\mu\nu})_{(ab)}(\sigma^{\rho\sigma})^{(ab)}A_\sigma=0$$

and one can obtain

$$A_\mu=\sigma^{a\dot{a}}_\mu\int d^4k\delta(k^2)e^{ik\cdot x}\left[\pi_a\tau_{\dot{a}}A_{-1}(k)+\widetilde{\tau}_a\widetilde{\pi}_{\dot{a}}A_1(k)\right].\tag{4.8}$$

[3] In the original paper by [12], the equation of motion for the Rarita-Schwinger field in the four component spinor notation reads $(\gamma\cdot\partial\Box\delta_{\mu\nu}-\gamma\cdot\partial\partial_\mu\partial_\nu-\frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\gamma_5\gamma^\sigma\partial^\rho\Box)\eta^\nu=0$ and one can convert this into the above two component spinor notation [15].

[4] One can also write this in different form $(\delta^{\nu\sigma}\Box-\partial^\nu\partial^\sigma)A_\sigma=0$ from the property of $\sigma^{\mu\nu}$ matrix [15].

For graviton $e_{\mu a\dot{a}}$, the equation of motion with fourth order derivative is given by [1]

$$\partial^{\dot{a}(c}\partial^{d)\dot{b}}\partial^{a(\dot{c}}\partial^{\dot{d})b}\sigma^{\mu}_{b\dot{b}}e_{\mu a\dot{a}} = 0$$

and the solution[5] is

$$e_{\mu a\dot{a}} = \sigma_{\mu}^{b\dot{b}}\int d^4k\delta(k^2)e^{ik\cdot x}\left[\pi_a\pi_b\tau_{\dot{a}}\tau_{\dot{b}}\left(e_{-2}(k)+i\frac{x_0}{k_0}e'_{-2}(k)\right)+\pi_{(a}\tilde{\tau}_{b)}\tau_{\dot{a}}\tau_{\dot{b}}e_{-1}(k)+\right.$$
$$\left.+\tilde{\tau}_a\tilde{\tau}_b\tilde{\pi}_{(\dot{a}}\tilde{\tau}_{\dot{b})}e_1(k)+\tilde{\tau}_a\tilde{\tau}_b\tilde{\pi}_{\dot{a}}\tilde{\pi}_{\dot{b}}\left(e_2(k)+i\frac{x_0}{k_0}e'_2(k)\right)\right]. \quad (4.9)$$

In this case, there exist two dipole fields of helicity -2 and 2. The linearized Weyl tensor $C_{\mu\nu\rho\sigma}$ can be defined through a Riemann tensor $R_{\mu\nu\rho\sigma}$ written in terms of second derivative of symmetric metric tensor, its contracted expressions $R_{\nu\sigma}, R_{\mu\sigma}, R_{\nu\rho}, R_{\mu\rho}$ and R^{μ}_{μ} [12] and C_{abcd} is the spinorial equivalent of this Weyl conformal tensor $C_{\mu\nu\rho\sigma}$: it is called the gravitational spinor in [18].

We will prove that the physical states of $\mathcal{N}=1$ conformal supergravity in four dimensions can be obtained from the spacetime fields described by the twistor fields $f^I(Z)$ and $g_I(Z)$. Let us start with the identification of the chiral superfield $\mathcal{W}^{\mathcal{N}=1}_{abc}(x,\theta_1)$ in the linearized conformal supergravity with the twistor fields of $\mathcal{N}=1$ as follows:

$$\mathcal{W}^{\mathcal{N}=1}_{abc}(x,\theta_1) = \left(\epsilon_{1ABC}D^A_aD^B_bD^C_c\int_{\mathbf{D}_{x,\theta}} g_I dZ^I\right)\Big|_{\theta_2=\theta_3=\theta_4=0},$$

where the Z^I are functions of λ^a for fixed x and θ through the twistor equations [2]

$$dZ^I = (d\lambda^a, d\mu^{\dot{a}}, d\psi^A) = (d\lambda^a, d\lambda_b x^{b\dot{a}}, d\lambda_c\theta^{cA})$$

and $\mathbf{D}_{x,\theta}$ is the curve with moduli x and θ. We will perform the action of superspace derivatives on the twistor field $g_I(Z)$. Here we used the twistor equations

$$(\mu^{\dot{a}}, \psi^A) = (x^{a\dot{a}}\lambda_a, \theta^{Aa}\lambda_a).$$

Let us write the twistor fields $g_I(Z)$ in terms of $\mathcal{W}^{\mathcal{N}=1}_{abc}(x,\theta_1)$ that has its physical fields, A_μ, η^a_μ and $e^{a\dot{a}}_\mu$. Our strategy here is to take $\mathcal{N}=4$ twistor fields and use the relation between our $\mathcal{N}=1$ chiral superfield and $\mathcal{N}=4$ superfield (4.2). Then the $\mathcal{N}=1$ twistor fields can be read off from the $\mathcal{N}=4$ twistor fields as we will see. As observed in section 3, we will see that the super derivative plays a role of changing $\mathrm{U}(1)_R$ charge. From the $\mathcal{N}=4$ twistor field results [1]

$$g^{\mathcal{N}=4}_{\dot{a}}(Z) = \cdots - i\frac{\lambda^a\sigma^0_{a\dot{a}}}{k^0}\left[(\psi^3)_A\hat{\eta}'^A_{-\frac{3}{2}} + (\psi^4)\hat{e}'_{-2}\right],$$

where the terms with lower ψ's than ψ^3 are not written explicitly, the relation turns out[6]

$$\epsilon_{1ABC}D^A_aD^B_bD^C_c\, g^{\mathcal{N}=4}_{\dot{a}}(Z)\Big|_{\theta_2=\theta_3=\theta_4=0} = -i\frac{\lambda^d\sigma^0_{d\dot{a}}}{k^0}\lambda_a\lambda_b\lambda_c\left[\hat{\eta}'_{-\frac{3}{2}} + \psi\hat{e}'_{-2}\right]. \quad (4.10)$$

[5]The conformal invariant equation for spin 2-field can be written as $\frac{1}{2}P^{\mu\lambda}_{2\rho\sigma}\Box^2 h_{\rho\sigma}$ with symmetric metric tensor $h_{\mu\nu}$ where $P^{\mu\nu}_{2\rho\sigma}$ is a spin 2 projector [3]: $P^{\mu\nu}_{2\rho\sigma} = \Pi^{\mu}_{(\rho}\Pi^{\nu}_{\sigma)} - \frac{1}{3}\Pi_{\rho\sigma}\Pi^{\mu\nu}$ and $\Pi^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \partial^{\mu}\Box^{-1}\partial_{\nu}$.

[6]One can compute this by using $(\psi^3)^D = \epsilon^{DEFG}\theta^e_E\theta^f_F\theta^g_G\lambda_e\lambda_f\lambda_g$ and $\psi^4 = \epsilon^{DEFG}\theta^d_D\theta^e_E\theta^f_F\theta^g_G\lambda_d\lambda_e\lambda_f\lambda_g$.

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Here $\hat{\eta}'_{-\frac{3}{2}}$ corresponding to $\hat{\eta}'^{A=1}_{-\frac{3}{2}}$ of $\mathcal{N}=4$ twistor field represents the twistor field of GL(1) charge -5 for the spacetime field $\eta'_{-\frac{3}{2}}(k)$ (4.7) of helicity $-3/2$ and $\hat{e}'_{-2}$ corresponding to $\hat{e}'_{-2}$ of $\mathcal{N}=4$ twistor field represents the twistor field of GL(1) charge -6 for the spacetime field $e'_{-2}(k)$ (4.9) of helicity -2. That is, for example, there exists a relation

$$\int d^4k\delta(k^2)e^{ik\cdot x}\eta'_{-\frac{3}{2}}(k) = \int d\lambda^a\lambda_a\hat{\eta}'_{-\frac{3}{2}} = \eta'_{-\frac{3}{2}}(x).$$

We also used the fact that there exists a twistor equation $\psi^{A=1} = \psi = \theta^{A=1,a}\lambda_a$. In section 3, the helicity and $\mathrm{U}(1)_R$ charge by the state $\partial_{\dot{a}}g^{\dot{a}}(Z)$ are given by $(0,1)$ and $(-\frac{1}{2},0)$. The product of $\lambda_a\lambda_b\lambda_c\hat{\eta}'_{-\frac{3}{2}}$ in (4.10) corresponds to $(0,1)$ twistor field while $\lambda_a\lambda_b\lambda_c\hat{e}'_{-2}$ corresponds to $(-\frac{1}{2},0)$ twistor field. The helicities are changed by the presence of three λ's (each of them has a helicity $\frac{1}{2}$). Therefore, the state $(-3/2,1)$ which shows an opposite helicity and an opposite $\mathrm{U}(1)_R$ charge from those described by $\mu^{\dot{a}}f_{\dot{a}}(Z)$ corresponds to $\hat{\eta}'_{-\frac{3}{2}}$ and the state $(-2,0)$ corresponds to $\hat{e}'_{-2}$.

Next, from the relation [1]

$$g_A^{\mathcal{N}=4}(Z) = \cdots + (\psi^2)_{[AB]}\hat{\eta}^B_{-\frac{1}{2}} + (\psi^2)_{[BC]}\hat{\xi}^{[BC]}_{-\frac{1}{2}A} + (\psi^3)_A\hat{e}_{-1} + (\psi^3)_B\hat{V}^B_{-1A} + (\psi^4)\hat{\bar{\eta}}_{-\frac{3}{2}A}$$

which was obtained by acting $\frac{\partial}{\partial\theta^A_a}$ on $\mathcal{W}^{\mathcal{N}=4}(x,\theta)$, one gets, by using two superspace derivatives rather than three because one superspace derivative is used at the level of $\mathcal{N}=4$ theory already,

$$\epsilon_{12BC}D^B_bD^C_c\, g^{\mathcal{N}=4}_{D=1}(Z)\Big|_{\theta_3=\theta_4=0} = \lambda_b\lambda_c\left[\hat{\eta}_{-\frac{1}{2}} + \psi\hat{A}_{-1} + \chi\hat{e}_{-2} + \psi\chi\hat{\bar{\eta}}_{-\frac{5}{2}}\right]. \qquad (4.11)$$

The χ can be written as $\lambda^a\lambda^b\lambda^c\theta^2_{abc}$ in general[7] and can be written as a linear combination of $\theta_i(i=2,3,4)$ with the coefficient that is cubic in λ's. In the above computation, we put $\theta_3=\theta_4=0$ at the end.

Let us describe how each term in (4.11) comes from those in $\mathcal{N}=4$ contents. The twistor field $\hat{\eta}_{-\frac{1}{2}}$ corresponding to $\hat{\xi}^{[B=3,C=4]}_{-\frac{1}{2},A=1}$ of $\mathcal{N}=4$ twistor field describes the twistor field of GL(1) charge -3 for the spacetime field $\eta_{-\frac{1}{2}}(k)$ of helicity $-\frac{1}{2}$. The contribution from $(\psi^2)_{[AB]}\hat{\eta}^B_{-\frac{1}{2}}$ when the SU(4) index A is equal to 1 vanishes because the two superspace derivatives in (4.11) pick up only the SU(4) indices 3 and 4. The twistor field $\hat{A}_{-1}$ corresponding to $\hat{V}^{B=2}_{-1,A=1}$ of $\mathcal{N}=4$ twistor field denotes the twistor field of GL(1) charge -4 for the spacetime field $A_{-1}(k)$ in (4.8) of helicity -1. The twistor field $\hat{e}_{-1}$ where $\hat{e}_{-2}$ multiplied by two λ's is equal to $\hat{e}_{-1}$ of $\mathcal{N}=4$ twistor field(Note that $\chi=\lambda_1\lambda_1\psi^2$ at

[7]One can express ψ and χ in terms of θ^a_A. One factorizes $\mathbf{CP}^{3|4}$ with respect to two other fermionic vector fields $N_1=\lambda_2\frac{\partial}{\partial\psi^2}-\lambda_1\frac{\partial}{\partial\psi^3}$ and $N_2=\lambda_1\frac{\partial}{\partial\psi^4}-\lambda_2\frac{\partial}{\partial\psi^3}$ where $\psi^A=\theta^{Aa}\lambda_a$. These vector fields allow us to reduce $\Pi\mathcal{O}(1)\oplus\Pi\mathcal{O}(1)\oplus\Pi\mathcal{O}(1)$ to $\Pi\mathcal{O}(3)$. Remember that the fermionic coordinates ψ and χ take values in the holomorphic line bundles $\mathcal{O}(1)$ and $\mathcal{O}(3)$ respectively [7, 19]. Here Π is a parity changing operator which changes the parity of the fiber coordinates when acting on a fiber bundle [20]. Then the fermionic coordinates invariant with respect to N_1 and N_2(i.e., they are annihilated by N_1 and N_2) of $\mathbf{WCP}^{3|2}$ are given by $\psi=\psi^1$ and $\chi=\lambda_1\lambda_1\psi^2+\lambda_1\lambda_2\psi^3+\lambda_2\lambda_2\psi^4\equiv\lambda_a\lambda_b\lambda_c\theta_2^{abc}$. We thank A.D. Popov for pointing out this.

$\theta_3 = \theta_4 = 0$) describes the twistor field of GL(1) charge -6 for the spacetime field $e_{-1}(k)$ of helicity -1. Here we used the same notation $\hat{e}_{-1}$ as $\mathcal{N} = 4$ field contents. Finally the twistor field $\hat{\overline{\eta}}_{-\frac{3}{2}}$ where the quantity $\hat{\overline{\eta}}_{-\frac{5}{2}}$ multiplied by two λ's corresponds to $\hat{\overline{\eta}}_{-\frac{3}{2},A=1}$ of $\mathcal{N} = 4$ twistor field denotes the twistor field of GL(1) charge -7 for the spacetime field $\overline{\eta}_{-\frac{3}{2}}(k)$ of helicity $-3/2$. The expression $\lambda_b \lambda_c \hat{\eta}_{-\frac{1}{2}}$ denotes the state $(\frac{1}{2}, 1)$ of section 3. Similarly $\lambda_b \lambda_c \hat{A}_{-1}$ corresponds to $(0,0)$ twistor field, $\lambda_b \lambda_c \hat{e}_{-2}$ does $(-1,0)$ twistor field and $\lambda_b \lambda_c \hat{\overline{\eta}}_{-\frac{5}{2}}$ describes $(-\frac{3}{2}, -1)$ twistor field. All these twistor fields have their counterparts in $f^{A=1}(Z)$ below that have opposite helicities and $\mathrm{U}(1)_R$ charges.

From the expression

$$g_a^{\mathcal{N}=4}(Z) = \cdots + \lambda_a \left[(\psi^3)_A \hat{\eta}^A_{-\frac{3}{2}} + (\psi^4) \hat{e}_{-2} \right],$$

one can compute

$$\epsilon_{1ABC} D_a^A D_b^B D_c^C \, g_d^{\mathcal{N}=4}(Z) \Big|_{\theta_2=\theta_3=\theta_4=0} = \lambda_d \lambda_a \lambda_b \lambda_c \left[\hat{\eta}_{-\frac{3}{2}} + \psi \hat{e}_{-2} \right]. \tag{4.12}$$

One can see that $\hat{\eta}_{-\frac{3}{2}}$ corresponding to $\hat{\eta}^{A=1}_{-\frac{3}{2}}$ of $\mathcal{N} = 4$ twistor field denotes the twistor field of GL(1) charge -5 for the spacetime field $\eta_{-\frac{3}{2}}(k)$ (4.7) of helicity $-3/2$ and $\hat{e}_{-2}$ corresponding to $\hat{e}_{-2}$ of $\mathcal{N} = 4$ twistor field denotes the twistor field of GL(1) charge -6 for the spacetime field $e_{-2}(k)$ (4.9) of helicity -2. Similarly, the state $(-\frac{3}{2}, 1)$ of section 3 corresponds to $\hat{\eta}_{-\frac{3}{2}}$(or $\lambda_a \lambda_b \lambda_c \hat{\eta}_{-\frac{3}{2}}$ has a helicity and $\mathrm{U}(1)_R$ charge by $(0,1)$) and the state $(-2, 0)$ corresponds to $\hat{e}_{-2}$. As observed above, one can see here the counterpart of these states in $\lambda^a f_a(Z)$ below.

One obtains the $f^I(Z)$ from the dual field $\widetilde{g}^I(\overline{Z})$ from the $\mathcal{N} = 4$ description:[8]

$$\begin{aligned}
\lambda^a f_a^{\mathcal{N}=4}(Z) \Big|_{\theta_2=\theta_3=\theta_4=0} &= \hat{e}'_2 + \psi \hat{\overline{\eta}}'_{\frac{3}{2}}, \\
f^{A=1}_{\mathcal{N}=4}(Z) \Big|_{\theta_3=\theta_4=0} &= \hat{\eta}_{\frac{3}{2}} + \psi \hat{e}_1 + \chi \hat{A}_0 + \psi \chi \hat{\overline{\eta}}_{-\frac{1}{2}}, \\
f_{\dot{a}}^{\mathcal{N}=4}(Z) \Big|_{\theta_2=\theta_3=\theta_4=0} &= \partial_{\dot{a}} \left(\hat{e}_2 + \psi \hat{\overline{\eta}}_{\frac{3}{2}} \right).
\end{aligned} \tag{4.13}$$

The helicity and $\mathrm{U}(1)_R$ charge for $f^I(Z)$ are specified in section 3. Their assignments coincide with those in (4.13). The twistor field $\hat{\overline{\eta}}'_{\frac{3}{2}}$ corresponds to $\hat{\overline{\eta}}'_{\frac{3}{2},A=1}$ of $\mathcal{N} = 4$ twistor field and $\psi^{A=1} = \psi$. After we put the $\theta_i = 0$ where $i = 2, 3, 4$, all the higher order terms in ψ^A(quadratic and so on) vanish. Now we move on the next twistor field description. Similarly, the $\hat{\eta}_{\frac{3}{2}}$ corresponds to $\hat{\eta}^{A=1}_{\frac{3}{2}}$ of $\mathcal{N} = 4$ twistor field. $\hat{A}_0$ multiplied by two λ's is equal to $\hat{V}^{B=2}_{1,A=1}$ of $\mathcal{N} = 4$ twistor field(Note that the χ is given by $\lambda_1 \lambda_1 \psi^2$ at $\theta_3 = \theta_4 = 0$ from previous footnote 7). The quantity $\hat{\overline{\eta}}_{-\frac{1}{2}}$ multiplied by two λ's corresponds to $\hat{\overline{\eta}}_{\frac{1}{2},B=2}$ of $\mathcal{N} = 4$ twistor field. Using the relation $\overline{\xi}^B_{[AB]} = 0$ [1], there is no such contribution from $(\psi^2)^{[BC]} \hat{\overline{\xi}}^A_{\frac{1}{2}[BC]}$. Of course, $\hat{e}_1$ stands for the same field as $\mathcal{N} = 4$ case. Finally $\hat{\overline{\eta}}_{\frac{3}{2}}$ corresponds to $\hat{\overline{\eta}}_{\frac{3}{2},A=1}$ of $\mathcal{N} = 4$ twistor field.

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[8] For convenience, we list here the $\mathcal{N} = 4$ description by [1] up to a quadratic in ψ^A: $\lambda^a f_a^{\mathcal{N}=4}(Z) = \hat{e}'_2 + \psi^A \hat{\overline{\eta}}'_{\frac{3}{2}A} + (\psi^2)_{[AB]} \hat{\overline{T}}_1'^{[AB]} + \cdots$, $f^A_{\mathcal{N}=4}(Z) = \hat{\eta}^A_{\frac{3}{2}} + \psi^B \hat{V}^B_{1A} + \psi^A \hat{e}_1 + (\psi^2)^{[BC]} \hat{\overline{\xi}}^A_{\frac{1}{2}[BC]} + (\psi^2)^{[AB]} \hat{\overline{\eta}}_{\frac{1}{2}B} + \cdots$, and $f_{\dot{a}}^{\mathcal{N}=4}(Z) = \partial_{\dot{a}} (\hat{e}_2 + \psi^A \hat{\overline{\eta}}_{\frac{3}{2}A} + (\psi^2)_{[AB]} \hat{\overline{T}}_1^{[AB]}) + \cdots$.

States in $\mathcal{N}=1$ CSG	$\mathrm{U}(1)_R$ charge	Helicity	Twistor superfields
η^a_μ	1	$-\frac{3}{2}'$	$\epsilon_{1ABC} D^A_a D^B_b D^C_c \; g_{\dot{a}}(Z)\Big\vert_{\theta_2=\theta_3=\theta_4=0}$
	1	$-\frac{3}{2}$	$\epsilon_{1ABC} D^A_a D^B_b D^C_c \; g_d(Z)\Big\vert_{\theta_2=\theta_3=\theta_4=0}$
	1	$-\frac{1}{2}$	$\epsilon_{12BC} D^B_b D^C_c \; g_{D=1}(Z)\Big\vert_{\theta_3=\theta_4=0}$
	1	$\frac{3}{2}$	$f^{A=1}(Z)\Big\vert_{\theta_3=\theta_4=0}$
A_μ	0	1	$f^{A=1}(Z)\Big\vert_{\theta_3=\theta_4=0}$
	0	-1	$\epsilon_{12BC} D^B_b D^C_c \; g_{D=1}(Z)\Big\vert_{\theta_3=\theta_4=0}$
$e^{a\dot{a}}_\mu$	0	2	$f_{\dot{a}}(Z)\Big\vert_{\theta_2=\theta_3=\theta_4=0}$
	0	$2'$	$\lambda^a f_a(Z)\Big\vert_{\theta_2=\theta_3=\theta_4=0}$
	0	1	$f^{A=1}(Z)\Big\vert_{\theta_3=\theta_4=0}$
	0	-1	$\epsilon_{12BC} D^B_b D^C_c \; g_{D=1}(Z)\Big\vert_{\theta_3=\theta_4=0}$
	0	-2	$\epsilon_{1ABC} D^A_a D^B_b D^C_c \; g_d(Z)\Big\vert_{\theta_2=\theta_3=\theta_4=0}$
	0	$-2'$	$\epsilon_{1ABC} D^A_a D^B_b D^C_c \; g_{\dot{a}}(Z)\Big\vert_{\theta_2=\theta_3=\theta_4=0}$
$\overline{\eta}^{\dot{a}}_\mu$	-1	$\frac{3}{2}$	$f_{\dot{a}}(Z)\Big\vert_{\theta_2=\theta_3=\theta_4=0}$
	-1	$\frac{3}{2}'$	$\lambda^a f_a(Z)\Big\vert_{\theta_2=\theta_3=\theta_4=0}$
	-1	$\frac{1}{2}$	$f^{A=1}(Z)\Big\vert_{\theta_3=\theta_4=0}$
	-1	$-\frac{3}{2}$	$\epsilon_{12BC} D^B_b D^C_c \; g_{D=1}(Z)\Big\vert_{\theta_3=\theta_4=0}$

Table 1: The $\mathrm{U}(1)_R$ charges, helicities of physical states in $\mathcal{N}=1$ conformal supergravity(CSG) in four dimensions. In the last column the relevant twistor superfields are given through (4.10)–(4.13) where $f^I(Z), g_I(Z)$ and D^A_a are $\mathcal{N}=4$ objects. There are four dipole fields (two states with the same helicity) of helicities $-2, -3/2, 3/2, 2$ in which the spacetime translations act in nondiagonalizable. The dependence on the fermionic coordinate χ of weight 3 of $\mathbf{WCP}^{3|2}$ appears only in both two superspace derivatives acting on $g_{A=1}(Z)$ and $f^{A=1}(Z)$ at $\theta_3=\theta_4=0$.

5. Concluding remarks

In this paper, the spectrum of massless fields in spacetime described by the twistor fields are summarized by two equations (3.1) and (3.2). By realizing the relation (4.2) between $\mathcal{N}=4$ chiral superfield and $\mathcal{N}=1$ chiral superfield, the physical states of $\mathcal{N}=1$ conformal supergravity are contained in the twistor superfields and they are summarized by the equations (4.10)–(4.13). We also put some results and present their relations in both sides in the table 1.

It would be interesting to see the physical states of $\mathcal{N}=2$ conformal supergravity from $\mathcal{N}=4$ superspace approach in terms of the spacetime fields described by the twistor superfields. For $\mathcal{N}=2$ conformal supergravity, the theory can be described off-shell by a chiral field strength $\mathcal{W}^{AB}_{ab}(x, \theta^A_a)$ where $A, B = 1, 2$ and the field contents [11] for this theory have the extra one antisymmetric tensor field, two spinors and three SU(2) gauge fields as well as those in $\mathcal{N} = 1$ conformal supergravity. One can think of topological B-model of $\mathbf{WCP}^{3|4}(1,1,1,1|1,1,p,q)$ where $p+q=2, p \neq 1, q \neq 1$ for the condition for Calabi-Yau supermanifold. For example, when $p=-1$ and $q=3$, this is the projective space with four bosonic homogeneous coordinates $Z^I, I=1,\ldots,4$ of weights one, and four fermionic homogeneous coordinates $\psi, \chi, \alpha, \beta$ of weights one, one, minus one, and three. The homogeneous coordinates have the following equivalence relation $(Z^I, \psi, \chi, \alpha, \beta) \approx (tZ^I, t\psi, t\chi, t^{-1}\alpha, t^3\beta)$ for t is an element of $\mathbf{C}^*$. This supermanifold should admit $\mathcal{N}=2$ superconformal symmetry SU(2, 2|2) acting on Z^I, ψ and χ.

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